$N_T = 8$, $D = 2$ Hodge–type cohomological gauge theory with global $SU(4)$ symmetry[1]

B. Geyer[a] [2] and D. Mülsch[b] [3]

[a] *Universität Leipzig, Naturwissenschaftlich-Theoretisches Zentrum and Institut für Theoretische Physik, D–04109 Leipzig, Germany*

[b] *Wissenschaftszentrum Leipzig e.V., D–04103 Leipzig, Germany*

Abstract

We show that the partially topological twisted $N = 16$, $D = 2$ super Yang–Mills theory gives rise to a $N_T = 8$ Hodge–type cohomological gauge theory with global $SU(4)$ symmetry.

1. Introduction

Some very enlightening, but preliminary attempts have been made to incorporate into the gauge–fixing procedure of general gauge theories besides the basic ingredience of BRST cohomology Ω also a co–BRST cohomology ${}^\star\Omega$ which, together with the BRST Laplacian W, form the same kind of superalgebra as the de Rham cohomology operators in differential geometry (for a review, see, e.g., Ref. [1]). This would allow, according to the Hodge–type decomposition $\psi = \omega + \Omega\chi + {}^\star\Omega\phi$ of a general quantum state, by imposing both the BRST condition $\Omega\psi = 0$ and the co–BRST condition ${}^\star\Omega\psi = 0$ upon ψ, to select the uniquely determined harmonic state ω thereby projecting onto the subspace of physical states.

It has been a long–standing problem to present a non–abelian field theoretical model obeying such a Hodge–type cohomological structure. Recently, the authors have shown [2] that the dimensional reduced Blau–Thompson model [3] — the novel $N_T = 2$ topological twist of the $N = 4$, $D = 3$ super Yang–Mills theory (SYM) — gives a prototype example of a $N_T = 4$, $D = 2$ Hodge–type cohomological gauge theory. The conjecture, that topological gauge theories could be possible candidates for Hodge–type cohomological theories was already asserted by van Holten [4]. In fact, $D = 2$ topological gauge theories [5] are of particular interest because of their relation to $N = 2$ superconformal theories [6] and Calabi–Yau moduli spaces [7].

Here we present another example of a Hodge–type cohomological gauge theory. It is obtained by a $N_T = 8$ topological twist of the Euclidean $N = 16$, $D = 2$ SYM, and its action localizes onto the moduli space of complexified flat connections. The $N_T = 8$ scalar supercharges Q^α and ${}^\star Q^\alpha$ of that theory form a topological superalgebra which is completely analogous to the de Rham cohomology. Both supercharges are interrelated by a discrete Hodge–type $\star$ operation and generate the topological shift and co–shift symmetries. In accordance with the group theoretical description of some classes of topologically twisted low–dimensional supersymmetric world–volume theories [3], it is shown that this $N_T = 8$ cohomological theory has actually the global symmetry group $SU(4)$. Such effective low–energy world–volume theories appear quite naturally in the study of curved D–branes and D–brane instantons wrapping around supersymmetric cocycles for special Lagrangian submanifolds of Calabi–Yau n–folds (see, e.g., [8, 9, 3]).

The paper is organized as follows: In Sec. 2 we briefly describe the BRST complex of general gauge theories based on harmonic gauges. In Sec. 3 we obtain the Euclidean $N = 16$, $D = 2$

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[2] Email: geyer@itp.uni-leipzig.de

[3] Email: muelsch@informatik.uni-leipzig.de

SYM theory with R–symmetry group $SO(8)$ from the $N = 4$, $D = 4$ SYM via dimensional reduction to $D = 2$. In Sec. 4 we perform the partiall $N_T = 8$ topological twist of this SYM theory thereby getting the looked for $N_T = 8$ Hodge–type cohomological theory with global symmetry group $SU(4)$. A more detailed version will be presented elsewhere [10].

2. BRST complex and Hodge decomposition

In order to select uniquely the physical states from the ghost–extended quantum state space some attempts [1] have been made to incorporate into the gauge–fixing procedure of general gauge theories besides the BRST cohomology Ω also a co–BRST cohomology ${}^\star\Omega$ which, together with the BRST Laplacian W, obeys the following BRST–complex:

$$\Omega^2 = 0, \qquad {}^\star\Omega^2 = 0, \qquad [\Omega, W] = 0, \qquad [{}^\star\Omega, W] = 0, \qquad W = \{\Omega, {}^\star\Omega\} \neq 0,$$

where Ω and ${}^\star\Omega$ have ghost number $+1$ and -1, respectively. Obviously, ${}^\star\Omega$ can not be identified with the anti–BRST operator $\bar{\Omega}$ which anticommutes with Ω.

Representations of this algebra for the first time have been considered by Nishijima [11]. However, since Ω and ${}^\star\Omega$ are nilpotent hermitian operators they cannot be realized in a Hilbert space. Instead, the BRST complex has to be represented in a Krein space $\mathcal{K}$ [12]. $\mathcal{K}$ is obtained from a Hilbert space $\mathcal{H}$ with non–degenerate positive inner product (χ, ψ) if $\mathcal{H}$ will be endowed also with a self–adjoint metric operator $J \neq 1$, $J^2 = 1$, allowing for the introduction of another non–degenerate, but indefinite scalar product $\langle\chi|\psi\rangle := (\chi, J\psi)$. With respect to the inner product Ω and ${}^\star\Omega = \pm J\Omega J$ are adjoint to each other, $(\chi, {}^\star\Omega\psi) = (\Omega\chi, \psi)$, however they are self–adjoint with respect to the indefinite scalar product of $\mathcal{K}$. Notice, that different inner products (χ, ψ) lead to different co–BRST operators!

From these definitions one obtains a remarkable correspondence between the BRST cohomology and the de Rham cohomology:

BRST operator	Ω,	differential	d,
co–BRST operator	${}^\star\Omega = \pm J\Omega J$,	co–differential	$\delta = \pm \star d \star$,
duality operation	J,	Hodge star	$\star$,
BRST Laplacian	$W = \{\Omega, {}^\star\Omega\}$,	Laplacian	$\Delta = \{d, \delta\}$.

Because of this correspondence one denotes a state ψ to be BRST (co–)closed iff $\Omega\psi = 0$ (${}^\star\Omega\psi = 0$), BRST (co–)exact iff $\psi = \Omega\chi$ ($\psi = {}^\star\Omega\phi$) and BRST harmonic iff $W\psi = 0$. Completely analogous to the Hodge decomposition theorem in differential geometry there exists a corresponding decomposition of any state ψ into a harmonic, an exact and a co–exact state, $\psi = \omega + \Omega\chi + {}^\star\Omega\phi$. The physical properties of ψ lie entirely within the BRST harmonic part ω which is given by the zero modes of the operator W; thereby $W\omega = 0$ implies $\Omega\omega = 0 = {}^\star\Omega\omega$, and vice versa. The cohomologies of the (co–)BRST operator are given by equivalence classes:

$$\mathrm{H}(\Omega) = \frac{\mathrm{Ker}\,\Omega}{\mathrm{Im}\,\Omega}, \qquad \psi \sim \psi' = \psi + \Omega\chi \quad \text{(equivalence class)},$$
$$\mathrm{H}({}^\star\Omega) = \frac{\mathrm{Ker}\,{}^\star\Omega}{\mathrm{Im}\,{}^\star\Omega}, \qquad \psi \sim \psi' = \psi + {}^\star\Omega\phi \quad \text{(equivalence class)}.$$

By imposing only the BRST gauge condition, $\Omega\psi = 0$, within the equivalence class of BRST–closed states $\psi = \omega + \Omega\chi$ besides the harmonic state ω there occur also spurious BRST–exact states, $\Omega\chi$, which have zero physical norm. On the other hand, by imposing also the co–BRST gauge condition, ${}^\star\Omega\psi = 0$, one gets for each BRST cohomology class the uniquely determined harmonic state, $\psi = \omega$.

3. Dimensional reduction of the $N=4$, $D=4$ super Yang–Mills theory

Our final aim is to show that by a partial topological twist of $N=16$, $D=2$ SYM one gets a $N_T=8$ Hodge–type cohomological theory with global symmetry group $SU(4)$. However, since the relationship between the twisted and untwisted fields is rather complex, let us first introduce the $N=16, D=2$ SYM. This theory can be obtained by dimensional reduction to $D=2$ from either $N=1$, $D=10$ SYM or $N=4$, $D=4$ SYM. Because the latter theory is well known, we choose the last possibility.

The field content of $N=4$, $D=4$ SYM consists of an anti–hermitean gauge field A_μ, two Majorana spinors $\lambda_{A\alpha}$ and $\bar{\lambda}_{\dot{A}}{}^{\alpha}$ ($\alpha=1,2,3,4$) which transform as the fundamental and its complex conjugate representation of $SU(4)$, respectively, and a set of complex scalar fields $G_{\alpha\beta}=\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}G^{\gamma\delta}$, which transform as the second–rank complex selfdual representation of $SU(4)$. All the fields take their values in the Lie algebra $Lie(\mathcal{G})$ of some compact gauge group $\mathcal{G}$.

In Euclidean space this theory has the following invariant action [13]:

$$
\begin{aligned}
S^{(N=4)} = \int_E d^4x \, \mathrm{tr}\Big\{ &\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar{\lambda}_{\dot{A}}{}^{\alpha}(\sigma_\mu)^{\dot{A}B}D^\mu\lambda_{B\alpha} + \tfrac{1}{64}[G_{\alpha\beta},G_{\gamma\delta}][G^{\alpha\beta},G^{\gamma\delta}] \\
&- \tfrac{1}{2}i\lambda_{A\alpha}[G^{\alpha\beta},\lambda^{A}{}_{\beta}] - \tfrac{1}{2}i\bar{\lambda}^{\dot{A}\alpha}[G_{\alpha\beta},\bar{\lambda}_{\dot{A}}{}^{\beta}] + \tfrac{1}{8}D_\mu G_{\alpha\beta}D^\mu G^{\alpha\beta}\Big\}, \qquad (1)
\end{aligned}
$$

where the numerically invariant tensors $(\sigma_\mu)^{A\dot{B}}$ and $(\sigma_\mu)_{\dot{A}B}$ are the Clebsch–Cordon coefficients relating the representation $(1/2,1/2$ of $SL(2,C)$ to the the vector representation of $SO(4)$,,

$$
\begin{aligned}
&(\sigma_\mu)^{\dot{A}B} = (-i\sigma_1,-i\sigma_2,-i\sigma_3,I_2), \qquad &&(\sigma_\mu)_{\dot{A}B} \equiv (\sigma_\mu)^{\dot{C}D}\epsilon_{\dot{C}\dot{A}}\epsilon_{DB} = (\sigma^*_\mu)^{\dot{A}B}, \\
&(\sigma_\mu)_{A\dot{B}} = (i\sigma_1,i\sigma_2,i\sigma_3,I_2), &&(\sigma_\mu)^{A\dot{B}} \equiv \epsilon^{AC}\epsilon^{\dot{B}\dot{D}}(\sigma_\mu)_{C\dot{D}} = (\sigma^*_\mu)_{A\dot{B}}, \qquad (2)
\end{aligned}
$$

$(\sigma_\mu)_{\dot{A}B}$ and $(\sigma_\mu)^{A\dot{B}}$ being the corresponding complex conjugate coefficients, respectively. Here, σ_a $(a=1,2,3)$ are the Pauli matrices. The selfdual and anti–selfdual generators of the $SO(4)$ rotations, $(\sigma_{\mu\nu})_{AB}$ and $(\sigma_{\mu\nu})_{\dot{A}\dot{B}}$, obey the relations

$$
\begin{aligned}
&(\sigma_\mu)^{A\dot{C}}(\sigma_\nu)_{\dot{C}}{}^{B} = (\sigma_{\mu\nu})^{AB} - \delta_{\mu\nu}\epsilon^{AB}, \\
&(\sigma_\rho)^{A\dot{C}}(\sigma_{\mu\nu})_{\dot{C}}{}^{\dot{B}} = \delta_{\rho\mu}(\sigma_\nu)^{A\dot{B}} - \delta_{\rho\nu}(\sigma_\mu)^{A\dot{B}} - \epsilon_{\mu\nu\rho\sigma}(\sigma^\sigma)^{A\dot{B}}, \qquad (3) \\
&(\sigma_\mu)_{\dot{A}C}(\sigma_\nu)^{C}{}_{\dot{B}} = (\sigma_{\mu\nu})_{\dot{A}\dot{B}} + \delta_{\mu\nu}\epsilon_{\dot{A}\dot{B}}, \\
&(\sigma_\rho)_{\dot{A}C}(\sigma_{\mu\nu})^{C}{}_{B} = \delta_{\rho\mu}(\sigma_\nu)_{\dot{A}B} - \delta_{\rho\nu}(\sigma_\mu)_{\dot{A}B} + \epsilon_{\mu\nu\rho\sigma}(\sigma^\sigma)_{\dot{A}B}. \qquad (4)
\end{aligned}
$$

The spinor index A (and analogously $\dot{A}$) is raised and lowered as follows: $\epsilon^{AC}\varphi_C{}^B=\varphi^{AB}$ and $\varphi_A{}^C\epsilon_{CB}=\varphi_{AB}$, where ϵ_{AB} (and analogous $\epsilon_{\dot{A}\dot{B}}$) is the invariant tensor of the group $SU(2)$, $\epsilon_{12}=\epsilon^{12}=\epsilon_{\dot{1}\dot{2}}=\epsilon^{\dot{1}\dot{2}}=1$.

The action (1) is manifestly invariant under hermitean conjugation:

$$
(A_\mu,\lambda_{A\alpha},\bar{\lambda}^{\dot{A}\alpha},G^{\alpha\beta}) \rightarrow (-A_\mu,\bar{\lambda}_{\dot{A}}{}^{\alpha},\lambda^{A}{}_{\alpha},G_{\alpha\beta}).
$$

Furthermore, making use of (3) and (4), one verifies that (1) is invariant also under the following on–shell supersymmetry transformations,

$$
\begin{aligned}
&Q_A{}^{\alpha}A_\mu = -i(\sigma_\mu)_{A\dot{B}}\bar{\lambda}^{\dot{B}\alpha}, \\
&Q_A{}^{\alpha}\bar{\lambda}_{\dot{B}}{}^{\beta} = (\sigma^\mu)_{A\dot{B}}D_\mu G^{\alpha\beta}, \\
&Q_A{}^{\alpha}G_{\beta\gamma} = 2i(\delta^{\alpha}{}_{\beta}\lambda_{A\gamma} - \delta^{\alpha}{}_{\gamma}\lambda_{A\beta}), \\
&Q_A{}^{\alpha}\lambda_{B\beta} = -\tfrac{1}{2}\delta^{\alpha}{}_{\beta}(\sigma^{\mu\nu})_{AB}F_{\mu\nu} - \tfrac{1}{2}\epsilon_{AB}[G^{\alpha\gamma},G_{\gamma\beta}]
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar{Q}_{\dot{A}\alpha} A_\mu = i(\sigma_\mu)_{\dot{A}B} \lambda^B{}_\alpha, \\
&\bar{Q}_{\dot{A}\alpha} \lambda_{B\beta} = (\sigma^\mu)_{\dot{A}B} D_\mu G_{\alpha\beta}, \\
&\bar{Q}_{\dot{A}\alpha} G^{\beta\gamma} = 2i(\delta_\alpha{}^\beta \bar{\lambda}_{\dot{A}}{}^\gamma - \delta_\alpha{}^\gamma \bar{\lambda}_{\dot{A}}{}^\beta), \\
&\bar{Q}_{\dot{A}\alpha} \bar{\lambda}_{\dot{B}}{}^\beta = -\tfrac{1}{2}\delta_\alpha{}^\beta (\sigma^{\mu\nu})_{\dot{A}\dot{B}} F_{\mu\nu} + \tfrac{1}{2}\epsilon_{\dot{A}\dot{B}} [G_{\alpha\gamma}, G^{\gamma\beta}].
\end{aligned}
$$

Let us recall that it is not possible to complete this superalgebra off–shell with a finite number of auxiliary fields [14].

In order to perform in (1) the dimensional reduction to $D = 2$ we re–name the third and fourth component of A_μ according to

$$
A_3 = \tfrac{1}{2}(\phi + \bar{\phi}), \qquad A_4 = \tfrac{1}{2} i (\phi - \bar{\phi}), \tag{5}
$$

reserving the notation A_μ ($\mu = 1, 2$) for the gauge field in $D = 2$. Moreover, we decompose the components of $(\sigma_\mu)_{\dot{A}}{}^B$, $(\sigma_{\mu\nu})_{\dot{A}}{}^{\dot{B}}$ and $(\sigma_\mu)_A{}^{\dot{B}}$, $(\sigma_{\mu\nu})_A{}^B$ in the following manner,

$$
\begin{aligned}
&(\sigma_\mu)_{\dot{A}}{}^B \rightarrow i(\sigma_\mu)_A{}^B, &&(\sigma_\mu)_A{}^{\dot{B}} \rightarrow i(\sigma_\mu)_A{}^B, \\
&(\sigma_3)_{\dot{A}}{}^B \rightarrow -i(\sigma_3)_A{}^B, &&(\sigma_3)_A{}^{\dot{B}} \rightarrow -i(\sigma_3)_A{}^B, \\
&(\sigma_4)_{\dot{A}}{}^B \rightarrow \delta_A{}^B, &&(\sigma_4)_A{}^{\dot{B}} \rightarrow -\delta_A{}^B, \\
&(\sigma_{\mu\nu})_{\dot{A}}{}^{\dot{B}} \rightarrow -i\epsilon_{\mu\nu}(\sigma_3)_A{}^B, &&(\sigma_{\mu\nu})_A{}^B \rightarrow i\epsilon_{\mu\nu}(\sigma_3)_A{}^B, \\
&(\sigma_{\mu 3})_{\dot{A}}{}^{\dot{B}} \rightarrow i\epsilon_{\mu\nu}(\sigma^\nu)_A{}^B, &&(\sigma_{\mu 3})_A{}^B \rightarrow -i\epsilon_{\mu\nu}(\sigma^\nu)_A{}^B, \\
&(\sigma_{\mu 4})_{\dot{A}}{}^{\dot{B}} \rightarrow i(\sigma_\mu)_A{}^B, &&(\sigma_{\mu 4})_A{}^B \rightarrow i(\sigma_\mu)_A{}^B, \\
&(\sigma_{34})_{\dot{A}}{}^{\dot{B}} \rightarrow -i(\sigma_3)_A{}^B, &&(\sigma_{34})_A{}^B \rightarrow -i(\sigma_3)_A{}^B,
\end{aligned} \tag{6}
$$

such that both the relations (3) and (4) become the algebra of the Pauli matrices,

$$
\begin{aligned}
&(\sigma_\mu)_A{}^C (\sigma_\nu)_{CB} = \delta_{\mu\nu}\epsilon_{AB} + i\epsilon_{\mu\nu}(\sigma_3)_{AB}, \qquad (\sigma_\mu, \sigma_3)_A{}^B = (\sigma_1, \sigma_2, \sigma_3), \\
&(\sigma_\mu)_A{}^C (\sigma_3)_{CB} = -i\epsilon_{\mu\nu}(\sigma^\nu)_{AB}, \\
&(\sigma_3)_A{}^C (\sigma_3)_{CB} = \epsilon_{AB}.
\end{aligned}
$$

Then, from (1) we obtain the Euclidean action of the $N = 16$, $D = 2$ SYM

$$
\begin{aligned}
S^{(N=16)} = \int_E d^2x \, \mathrm{tr} \Big\{ &\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} D_\mu \bar{\phi} D^\mu \phi - \tfrac{1}{8} [\bar{\phi}, \phi]^2 \\
&- \tfrac{1}{2} \bar{\lambda}_A{}^\alpha (\sigma_3)^{AB} [\phi + \bar{\phi}, \lambda_{B\alpha}] + \tfrac{1}{2} \bar{\lambda}^{A\alpha} [\phi - \bar{\phi}, \lambda_{A\alpha}] \\
&+ \bar{\lambda}_A{}^\alpha (\sigma_\mu)^{AB} D^\mu \lambda_{B\alpha} - \tfrac{1}{2} i \lambda_{A\alpha} [G^{\alpha\beta}, \lambda^A{}_\beta] - \tfrac{1}{2} i \bar{\lambda}^{A\alpha} [G_{\alpha\beta}, \bar{\lambda}_A{}^\beta] \\
&+ \tfrac{1}{8} D_\mu G_{\alpha\beta} \, D^\mu G^{\alpha\beta} + \tfrac{1}{8} [\bar{\phi}, G_{\alpha\beta}][\phi, G^{\alpha\beta}] + \tfrac{1}{64} [G_{\alpha\beta}, G_{\gamma\delta}][G^{\alpha\beta}, G^{\gamma\delta}] \Big\}.
\end{aligned} \tag{7}
$$

Since the decompositions (6) explicitly include various factors of i, the action (7) is no longer manifestly invariant under hermitean conjugation. Rather, it is invariant under the following Z_2 symmetry,

$$
Z_2: \qquad (A_\mu, \phi, \bar{\phi}, \lambda_{A\alpha}, \bar{\lambda}^{A\alpha}, G^{\alpha\beta}) \rightarrow (A_\mu, \bar{\phi}, \phi, -\bar{\lambda}_A{}^\alpha, -\lambda^A{}_\alpha, G_{\alpha\beta}). \tag{8}
$$

Denoting the $N=16$ spinorial supercharges in $D=2$ by ${Q_A}^\alpha$ and $\bar{Q}_{A\alpha}$, which are interchanged by the Z_2 symmetry (8), the transformation rules of the re–named fields are:

$$
\begin{aligned}
&{Q_A}^\alpha A_\mu = (\sigma_\mu)_{AB} \bar{\lambda}^{B\alpha}, \\
&{Q_A}^\alpha \phi = -(\sigma_3)_{AB} \bar{\lambda}^{B\alpha} - {\bar{\lambda}_A}^{\ \alpha}, \\
&{Q_A}^\alpha \bar{\phi} = -(\sigma_3)_{AB} \bar{\lambda}^{B\alpha} + {\bar{\lambda}_A}^{\ \alpha}, \\
&{Q_A}^\alpha {\bar{\lambda}_B}^{\ \beta} = \tfrac{1}{2} i (\sigma^\mu)_{AB} D_\mu G^{\alpha\beta} - \tfrac{1}{2} i (\sigma_3)_{AB} [\phi + \bar{\phi}, G^{\alpha\beta}] - \tfrac{1}{2} i \epsilon_{AB} [\phi - \bar{\phi}, G^{\alpha\beta}], \\
&{Q_A}^\alpha G_{\beta\gamma} = 2i(\delta^\alpha_{\ \beta} \lambda_{A\gamma} - \delta^\alpha_{\ \gamma} \lambda_{A\beta}), \\
&{Q_A}^\alpha \lambda_{B\beta} = \tfrac{1}{2} i \delta^\alpha_{\ \beta} \epsilon^{\mu\nu} (\sigma_\nu)_{AB} D_\mu (\phi + \bar{\phi}) + \tfrac{1}{2} \delta^\alpha_{\ \beta} (\sigma^\mu)_{AB} D_\mu (\phi - \bar{\phi}) \\
&\qquad\qquad + \tfrac{1}{2} \delta^\alpha_{\ \beta} (\sigma^3)_{AB} [\phi, \bar{\phi}] - \tfrac{1}{2} i \delta^\alpha_{\ \beta} \epsilon^{\mu\nu} (\sigma_3)_{AB} F_{\mu\nu} - \tfrac{1}{2} \epsilon_{AB} [G^{\alpha\gamma}, G_{\gamma\beta}].
\end{aligned}
\tag{9}
$$

4. $N_T = 8$ topological twist of the $N=16$, $D=2$ super Yang–Mills theory

Let us now perform the $N_T = 8$ topological twist of the $N=16$, $D=2$ SYM (for the group theoretical description of that topological twist we refer to [3]). For that purpose we introduce the following set of twisted fields: A $SU(4)$–quartet of Grassmann–odd vector fields ψ_μ^α, two $SU(4)$–quartets of Grassmann–odd scalar fields, $\bar{\eta}_\alpha$ and $\bar{\zeta}_\alpha$ which transform as the fundamental and its complex conjugate representation of $SU(4)$, respectively, and a $SU(4)$–sextet of Grassmann–even complex scalar fields $M_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta} M^{\gamma\delta}$, which transform as the second–rank complex selfdual representation of $SU(4)$.

Explicitly, the relationships between the original and the twisted fields are chosen as follows:

$$
\begin{aligned}
\lambda_{A\alpha} &= \tfrac{1}{2} \begin{pmatrix} i(\sigma^\mu)_{AB} (\psi_\mu^1 - \epsilon_{\mu\nu} \psi^{\nu 3}) + (\sigma_3)_{AB} (\bar{\eta}_4 + \bar{\zeta}_2) + i \epsilon_{AB} (\bar{\zeta}_4 - \bar{\eta}_2) \\ i(\sigma^\mu)_{AB} (\psi_\mu^2 + \epsilon_{\mu\nu} \psi^{\nu 4}) + (\sigma_3)_{AB} (\bar{\eta}_3 - \bar{\zeta}_1) + i \epsilon_{AB} (\bar{\zeta}_3 + \bar{\eta}_1) \end{pmatrix}, \\
\bar{\lambda}^{A\alpha} &= \tfrac{1}{2} \begin{pmatrix} i(\sigma^\mu)^{AB} (\epsilon_{\mu\nu} \psi^{\nu 4} - \psi_\mu^2) + (\sigma_3)^{AB} (\bar{\zeta}_1 + \bar{\eta}_3) - i \epsilon^{AB} (\bar{\eta}_1 - \bar{\zeta}_3) \\ i(\sigma^\mu)^{AB} (\epsilon_{\mu\nu} \psi^{\nu 3} + \psi_\mu^1) + (\sigma_3)^{AB} (\bar{\zeta}_2 - \bar{\eta}_4) - i \epsilon^{AB} (\bar{\eta}_2 + \bar{\zeta}_4) \end{pmatrix},
\end{aligned}
\tag{10}
$$

between $\lambda_{A\alpha}$, $\bar{\lambda}^{A\alpha}$ and the twisted vector and scalar fields ψ_μ^α, $\bar{\eta}_\alpha$, $\bar{\zeta}_\alpha$, as well as

$$
\begin{aligned}
\phi &= M_1 - i M_2, \\
\bar{\phi} &= M_1 + i M_2,
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
G_{\alpha\beta} &= \begin{pmatrix} \epsilon_{AB} M_6 & -(\sigma^\mu)_{AB} V_\mu + (\sigma_3)_{AB} M_3 - i \epsilon_{AB} M_4 \\ (\sigma^\mu)_{AB} V_\mu - (\sigma_3)_{AB} M_3 - i \epsilon_{AB} M_4 & \epsilon_{AB} M_5 \end{pmatrix}, \\
G^{\alpha\beta} &= \begin{pmatrix} \epsilon^{AB} M_5 & (\sigma^\mu)^{AB} V_\mu - (\sigma_3)^{AB} M_3 + i \epsilon^{AB} M_4 \\ -(\sigma^\mu)^{AB} V_\mu + (\sigma_3)^{AB} M_3 + i \epsilon^{AB} M_4 & \epsilon^{AB} M_6 \end{pmatrix},
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
M_1 &= \tfrac{1}{2}(M^{12} + M^{34}), \qquad & M_3 &= \tfrac{1}{2} i (M^{12} - M^{34}), \qquad & M_5 &= M^{24}, \\
M_2 &= \tfrac{1}{2}(M^{14} + M^{23}), \qquad & M_4 &= \tfrac{1}{2} i (M^{14} - M^{23}), \qquad & M_6 &= M^{31},
\end{aligned}
$$

between ϕ, $\bar{\phi}$, $G_{\alpha\beta}$ and the twisted vector and scalar fields V_μ and $M_{\alpha\beta}$, respectively.

Thereby, the assignment between the index α of the internal group and the spinor index A is the following: In (10) the spinor indices $B = 1, 2$ at the top and at the bottom of both columns correspond to the values $\alpha = 1, 2$ and $\alpha = 3, 4$ of both spinors $\lambda_{A\alpha}$ and $\bar{\lambda}^{A\alpha}$, respectively. Similary, in (12) the spinor indices $A = 1, 2$ (resp. $B = 1, 2$) at the upper and at the lower raw (resp. at the left and at the right column) of the both matrices correspond to the values $\alpha = 1,2$ and $\alpha = 3,4$ (resp. $\beta = 1,2$ and $\beta = 3,4$) of the scalar fields $G_{\alpha\beta}$, respectiely. By using the explicit form (2) of the Clebsch–Gordon coefficients one establishes that $G_{\alpha\beta}$ and $G^{\alpha\beta}$ in (12) are actually dual to each other, $G_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}G^{\gamma\delta}$.

The relationship between the spinorial supercharges $Q^{A\alpha}$ and $\bar{Q}_{A\alpha}$, being interrelated by the Z_2 symmetry (8), and the twisted scalar and vector supercharges Q^α, ${}^\star Q^\alpha$ and $\bar{Q}_{\mu\alpha}$, being interchanged by a discrete Hodge–type $\star$ operation (see Eq. (15) below), is quite similar to the ones of the spinor fields, Eq. (10), namely

$$
\begin{aligned}
Q^{A\alpha} &= \tfrac{1}{2}\begin{pmatrix} i(\sigma^\mu)^{AB}(\bar{Q}_{\mu 1} - \epsilon_{\mu\nu}\bar{Q}_3^\nu) - (\sigma_3)^{AB}(Q^4 - i\,{}^\star Q^2) - \epsilon^{AB}({}^\star Q^4 - iQ^2) \\ i(\sigma^\mu)^{AB}(\bar{Q}_{\mu 2} + \epsilon_{\mu\nu}\bar{Q}_4^\nu) - (\sigma_3)^{AB}(Q^3 + i\,{}^\star Q^1) - \epsilon^{AB}({}^\star Q^3 + iQ^1) \end{pmatrix}, \\
\bar{Q}_{A\alpha} &= \tfrac{1}{2}\begin{pmatrix} i(\sigma^\mu)_{AB}(\epsilon_{\mu\nu}\bar{Q}_4^\nu - \bar{Q}_{\mu 2}) + (\sigma_3)_{AB}(i\,{}^\star Q^1 - Q^3) + \epsilon_{AB}(iQ^1 - {}^\star Q^3) \\ i(\sigma^\mu)_{AB}(\epsilon_{\mu\nu}\bar{Q}_3^\nu + \bar{Q}_{\mu 1}) + (\sigma_3)_{AB}(i\,{}^\star Q^2 + Q^4) + \epsilon_{AB}(iQ^2 + {}^\star Q^4) \end{pmatrix}.
\end{aligned}
\tag{13}
$$

After performing in (7) the topological twist (10) – (12) and introducing the Grassmann–even auxiliary fields B, $\bar{B}$, Y and $E_{\mu\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}E_\mu^{\gamma\delta}$ one gets the following $N_T = 8$ Hodge–type cohomological gauge theory with global symmetry group $SU(4)$:

$$
\begin{aligned}
S^{(N_T=8)} = \int_E d^2x\, \mathrm{tr}\Big\{ &\tfrac{1}{4}i\epsilon^{\mu\nu}BF_{\mu\nu}(A+iV) - \tfrac{1}{4}i\epsilon^{\mu\nu}\bar{B}F_{\mu\nu}(A-iV) - \tfrac{1}{2}\bar{B}B \\
&- \epsilon^{\mu\nu}\bar{\zeta}_\alpha D_\mu(A+iV)\psi_\nu^\alpha - \bar{\eta}_\alpha D^\mu(A-iV)\psi_\mu^\alpha - \tfrac{1}{4}E^\mu_{\alpha\beta}E_\mu^{\alpha\beta} \\
&+ \tfrac{1}{2}i\epsilon^{\mu\nu}M_{\alpha\beta}\{\psi_\mu^\alpha, \psi_\nu^\beta\} + iM^{\alpha\beta}\{\bar{\eta}_\alpha, \bar{\zeta}_\beta\} - YD^\mu(A)V_\mu - \tfrac{1}{2}Y^2 \\
&+ \tfrac{1}{8}D^\mu(A+iV)M_{\alpha\beta}\; D_\mu(A-iV)M^{\alpha\beta} + \tfrac{1}{64}[M_{\alpha\beta}, M_{\gamma\delta}][M^{\alpha\beta}, M^{\gamma\delta}]\Big\}.
\end{aligned}
\tag{14}
$$

In this $SU(4)$ symmetric form the action (14) is manifestly inariant under the following Hodge–type $\star$ symmetry, defined by the replacements

$$
\varphi \equiv \begin{bmatrix} \partial_\mu & A_\mu & V_\mu & \\ \psi_\mu^\alpha & \bar{\eta}_\alpha & \bar{\zeta}_\alpha & M^{\alpha\beta} \\ B & \bar{B} & Y & E_\mu^{\alpha\beta} \end{bmatrix} \quad \Rightarrow \quad \star\varphi = \begin{bmatrix} \epsilon_{\mu\nu}\partial^\nu & \epsilon_{\mu\nu}A^\nu & -\epsilon_{\mu\nu}V^\nu & \\ -i\psi_\mu^\alpha & -i\bar{\zeta}_\alpha & i\bar{\eta}_\alpha & -M^{\alpha\beta} \\ -\bar{B} & -B & -Y & \epsilon_{\mu\nu}E^{\nu\alpha\beta} \end{bmatrix}.
\tag{15}
$$

with the property $\star(\star\varphi) = -P\varphi$. Here, P is the operator of Grassmann–parity whose eigenalues are defined by

$$
P\varphi = \begin{cases} +\varphi & \text{if } \varphi \text{ is Grassmann-odd,} \\ -\varphi & \text{if } \varphi \text{ is Grassmann-even} \end{cases}.
$$

Hence, after twisting the Z_2 symmetry (8) changes into the Hodge–type $\star$ symmetry (15).

The transformations rules for the topological shift symmetry, generated by Q^α, are

$$
\begin{aligned}
&Q^\alpha A_\mu = \psi_\mu^\alpha, \\
&Q^\alpha V_\mu = -i\psi_\mu^\alpha, \\
&Q^\alpha M_{\beta\gamma} = 2i(\delta^\alpha{}_\beta\bar{\zeta}_\gamma - \delta^\alpha{}_\gamma\bar{\zeta}_\beta), \\
&Q^\alpha\psi_\mu^\beta = E_\mu^{\alpha\beta} - i\epsilon_{\mu\nu}D^\nu(A-iV)M^{\alpha\beta}, \\
&Q^\alpha\bar{\zeta}_\beta = i\delta^\alpha{}_\beta B, \\
&Q^\alpha B = 0, \\
&Q^\alpha\bar{\eta}_\beta = i\delta^\alpha{}_\beta Y + \tfrac{1}{2}[M^{\alpha\gamma}, M_{\gamma\beta}], \\
&Q^\alpha Y = [M^{\alpha\beta}, \bar{\zeta}_\beta], \\
&Q^\alpha\bar{B} = -2[M^{\alpha\beta}, \bar{\eta}_\beta], \\
&Q^\alpha E^\mu_{\beta\gamma} = \delta^\alpha{}_{[\beta}\big(\epsilon^{\mu\nu}D_\nu(A+iV)\bar{\zeta}_{\gamma]} - D^\mu(A-iV)\bar{\eta}_{\gamma]} - i\epsilon^{\mu\nu}[M_{\gamma]\delta}, \psi_\nu^\delta]\big).
\end{aligned}
\tag{16}
$$

From combining Q^α with the above displayed Hodge–type $\star$ symmetry one gets the corresponding transformations rules for the topological co–shift symmetry: ${}^\star Q^\alpha = P \star Q^\alpha \star$.

By a straightforward calculation one verifies that both the supercharges Q^α and ${}^\star Q^\alpha$ provide an *off–shell* realization of the following topological superalgebra,

$$\{Q^\alpha, Q^\beta\} = 0, \qquad \{Q^\alpha, {}^\star Q^\beta\} = -2\delta_G(M^{\alpha\beta}), \qquad \{{}^\star Q^\alpha, {}^\star Q^\beta\} = 0, \tag{17}$$

where the field–dependent gauge transformations are defined by $\delta_G(M^{\alpha\beta})A_\alpha = -D_\alpha M^{\alpha\beta}$ and $\delta_G(M^{\alpha\beta})X = [M^{\alpha\beta}, X]$ for all the other fields.

Obviously, the structure of this superalgebra is directly analogous to the de Rham cohomology in differential geometry: The exterior and the co–exterior derivatives d and $\delta = \pm \star d \star$, being interrelated by the duality $\star$ operation, correspond to the nilpotent topological shift and co–shift operators Q^α and ${}^\star Q^\alpha = P \star Q^\alpha \star$, respectively. Moreover, the Laplacian $\Delta = \{d, \delta\}$ corresponds to the field–dependent gauge generator $\delta_G(M^{\alpha\beta})$, so that we have indeed a perfect example of a Hodge–type cohomological gauge theory.

Furthermore, by an explicit calculation one can verify that the action (14) is also invariant under the following *on–shell* vector supersymmetries,

$$\begin{aligned}
&\bar{Q}_{\mu\alpha} A_\nu = \delta_{\mu\nu} \bar{\eta}_\alpha - \epsilon_{\mu\nu} \bar{\zeta}_\alpha, \\
&\bar{Q}_{\mu\alpha} V_\nu = -i\delta_{\mu\nu} \bar{\eta}_\alpha - i\epsilon_{\mu\nu} \bar{\zeta}_\alpha, \\
&\bar{Q}_{\mu\alpha} M^{\beta\gamma} = 2i\epsilon_{\mu\nu}(\delta_\alpha^{~\beta} \psi^{\nu\gamma} - \delta_\alpha^{~\gamma} \psi^{\nu\beta}), \\
&\bar{Q}_{\mu\alpha} \bar{\zeta}_\beta = \epsilon_{\mu\nu} E^\nu_{\alpha\beta} + iD_\mu(A - iV)M_{\alpha\beta}, \\
&\bar{Q}_{\mu\alpha} \bar{\eta}_\beta = E_{\mu\alpha\beta} + i\epsilon_{\mu\nu} D^\nu(A + iV)M_{\alpha\beta}, \\
&\bar{Q}_{\mu\alpha} \psi^\beta_\nu = -2\delta_\alpha^{~\beta} F_{\mu\nu}(A) - 2i\delta_\alpha^{~\beta} D_\mu(A)V_\nu - i\delta_\alpha^{~\beta} \delta_{\mu\nu} Y - i\delta_\alpha^{~\beta} \epsilon_{\mu\nu} \bar{B} + \tfrac{1}{2}\delta_{\mu\nu}[M_{\alpha\gamma}, M^{\gamma\beta}], \\
&\bar{Q}_{\mu\alpha} \bar{B} = 2i\epsilon_{\mu\nu} D^\nu(A + iV)\bar{\eta}_\alpha, \\
&\bar{Q}_{\mu\alpha} Y = 2iD_\mu(A - iV)\bar{\eta}_\alpha - \epsilon_{\mu\nu}[M_{\alpha\beta}, \psi^{\nu\beta}], \\
&\bar{Q}_{\mu\alpha} B = 2i\epsilon_{\mu\nu} D^\nu(A - iV)\bar{\eta}_\alpha + 4iD_\mu(A)\bar{\zeta}_\alpha + 2[M_{\alpha\beta}, \psi^\beta_\mu], \\
&\bar{Q}_{\mu\alpha} E^{\beta\gamma}_\nu = -\delta_\alpha^{~[\beta} D_{[\mu}(A + iV)\psi_{\nu]}^{~\gamma]} - \delta_\alpha^{~[\beta} \delta_{\mu\nu} D^\rho(A - iV)\psi^{\gamma]}_\rho + i\delta_\alpha^{~[\beta}[M^{\gamma]\delta}, \epsilon_{\mu\nu}\bar{\eta}_\delta - \delta_{\mu\nu}\bar{\zeta}_\delta].
\end{aligned} \tag{18}$$

In addition, this action is also invariant under the co–vector supersymmetries

$${}^\star \bar{Q}_{\mu\alpha} = P \star \bar{Q}_{\mu\alpha} \star \doteq i\bar{Q}_{\mu\alpha},$$

which on–shell, i.e., by using *only* the equations of motion of the auxiliarly fields, become i times the vector supersymmetries! Hence, it holds

$$(Q^\alpha, {}^\star Q^\alpha, \bar{Q}_{\mu\alpha}) S^{(N_T=8)} = 0,$$

and the total number of (real) supercharges is actually $N = 16$.

Finally, let us mention that there is also a $N_T = 4$ topological twist of $N = 16$, $D = 2$ SYM with global symmetry group $SO(4) \otimes SU(2)$. This topological theory can be regarded as the $N_T = 4$ super–BF theory coupled to a spinorial hypermultiplet. Another way of obtaining the action of this theory is to dimensionally reduce either the higher dimensional analogue of the Donaldson–Witten theory in $D = 8$ [15, 16] to $D = 2$ or to dimensionally reduce the $N_T = 1$ half–twisted theory [17] in $D = 4$ to $D = 2$. However, that topological twist does not lead to another Hodge–type cohomological theory, since the underlying cohomology is only equivariantly nilpotent and not strictly nilpotent as in Eqs. (17).

References

[1] J. W. van Holten, *Aspects of BRST quantization*, Lectures at Summer School "Geometry and Topology in Physics", Rot a.d. Rot (Germany), Sept. 2001 (hep-th/0201124), and references therein

[2] B. Geyer and D. Mülsch, *Phys. Lett.* **B 518** (2001) 181; *Hodge–type cohomological gauge theories*, Proc. 5. Int. A. Friedman Seminar, Joao Pessoa, April 2002, to appear: Int. J. Mod. Phys. A

[3] M. Blau and G. Thompson, *Nucl. Phys.* **B 492** (1997) 545

[4] J. W. van Holten, *Phys. Rev. Lett.* **64** (1990) 2863

[5] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys.* **B 352** (1991) 59; E. Verlinde and N. P. Warner, *Phys. Lett.* **B 269** (1991) 96

[6] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. del Giudice, P. di Vecchia, S. Ferrara, F. Gliozzi, R. Musto and R. Pettorino, *Phys. Lett.* **B 62** (1976) 105; P. Gepner, *Nucl. Phys.* **B 296** (1988) 757; W. Lerche, C. Vafa and N. P. Warner, *Nucl. Phys.* **B 324** (1989) 427; B. Green, C. Vafa and N. P. Warner, *Nucl. Phys.* **B 324** (1989) 371

[7] P. Candelas, C. T. Horowitz, A. Strominger and E. Witten, *Nucl. Phys.* **B 258** (1985) 46; P. Candelas, A. M. Dale, C. A. Lutken and R. Schimmrigk, *Nucl. Phys.* **B 298** (1988) 493; P. Candelas, *Nucl. Phys.* **B 298** (1989) 458; P. Candelas and X. de la Ossa, *Nucl. Phys.* **B 342** (1990) 246

[8] E. Witten, *Nucl. Phys.* **B 323** (1989) 113

[9] M. Bershadsky, V. Sadov and C. Vafa, *Nucl. Phys.* **B 448** (1995) 166

[10] B. Geyer and D. Mülsch, *Higher dimensional analogue of the Blau–Thompson model and* $N_T = 8$ *Hodge–type cohomological gauge theories*, to appear

[11] K. Nishijima, *Prog. Theor. Phys.* **80** (1988) 897; **80** (1988) 905

[12] A.V. Razumov and G.N. Rybkin *Nucl. Phys.* **B 332** (1990) 209

[13] L. Brink, J. Schwarz and J. Scherk, *Nucl. Phys.* **B 121** (1977) 77

[14] W. Siegel and M. Rocek, *Phys. Lett.* **B 105** (1981) 275; V. Rivelles and J. G. Taylor, *J. Phys. A: Math. Gen.* **A 15** (1982) 163

[15] L. Baulieu, H. Kanno and I. Singer, *Commun. Math. Phys.* **194** (1998) 149; *Cohomological Yang–Mills theory in eight dimensions*, in: *Dualities in String Theory*, Seoul 1997 , p. 365 (hep-th/9705127)

[16] B. S. Acharya and M. O'Loughlin, *Phys. Rev.* **D 55** (1997) 4521; B. S. Acharya, M. O'Loughlin and B. Spence, *Nucl. Phys.* **B 503** (1997) 657

[17] J. Yamron, *Phys. Lett.* **B 213** (1988) 325